SOLUTION TO PROBLEM #12375

Problem #12375. Proposed by H. Chen (USA). Let

$$I_n = \int_0^\infty \left(1 - x^2 \sin^2\left(\frac{1}{x}\right)\right)^n dx.$$

Prove that I_n is a rational multiple of π whenever n is a positive integer.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Change variables $t = \frac{1}{x}, dx = -\frac{dt}{t^2}$, recall $\operatorname{sinc}(x) = \frac{\sin x}{x}$ and convert the given integral into

$$I_n = \int_0^\infty \left(1 - \operatorname{sinc}^2 t\right)^n \frac{dt}{t^2}.$$

Introduce the functions $f(t) = 1 - \operatorname{sinc}^2 t, g(t) = \frac{1-\operatorname{sinc}^2 t}{t^2}, F(s) = 2\pi\delta_0(s) + \frac{\pi(|s|-2)}{2} \cdot \chi_{|s|<2}(s)$ and $G(s) = -\frac{\pi(|s|-2)^3}{12} \cdot \chi_{|s|<2}(s)$, where $\chi_{|s|<2}$ stands for the *characteristic function* of the interval $\{|s| < 2\}$. Define the *Fourier transform* by $(\mathcal{F}h)(s) = \int_{\mathbb{R}} h(t)e^{-its}dt$. Then, it is an easy check that $\frac{1}{2\pi}\mathcal{F}^{-1}F = f$ and $\frac{1}{2\pi}\mathcal{F}^{-1}G = g$. Therefore, $\mathcal{F}f = F, \mathcal{F}(g) = G, fg = \mathcal{F}^{-1}(F * G)$. Since all our functions are *even,* the inversion theorem shows $\mathcal{F}(fg) = \frac{1}{2\pi}(F * G)$, as a *convolution.* An obvious repetition implies that

$$\mathcal{F}(f^{n-1}g) = \frac{1}{2\pi}(F * \cdots * G) = \frac{1}{(2\pi)^{n-1}}(F *^{n-1} G).$$

If $n = 1$ then $\frac{2\pi}{3} = G(0) = \mathcal{F}(g)(0) = 2I_1$; in general, $\frac{1}{(2\pi)^{n-1}}(F *^{n-1} G)(0) = \mathcal{F}(f^{n-1}g)(0) = 2I_n$. Since both $\frac{1}{2\pi}F$ and $\frac{1}{2\pi}G$ are polynomials in $\mathbb{Q}[s]$, we realize $\frac{1}{(2\pi)^n}(F *^{n-1} G) \in \mathbb{Q}[s]$. That means exactly one π is spared in I_n so that $\frac{1}{\pi}I_n \in \mathbb{Q}[s]$. The proof is complete. □