SOLUTION TO PROBLEM #12385

Problem #12385. Proposed by H. Ohtsuka (Japan). Let n be a positive integer. Prove

$$\sum_{1\le i\le k\le n} \frac{(-2)^k}{k+1}\binom{n}{k}\binom{k}{i}^{-1} = \frac{(-1)^n-1}{2n}.$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Start with the inner sum on the left-hand side. Define $F(k,i):=\frac{2^{k+1}}{(k+1)\binom{k}{i}}$ and $G(k,i):=-F(k+1,i)$, geared up to apply the Wilf-Zeilberger method. Check that $F(k+1,i)-F(k,i)=G(k,i+1)-G(k,i)$ is satisfied. Then, sum both sides over $1\le i\le k$ and observe that $\sum_{i=1}^k G(k,i+1)-\sum_{i=1}^k G(k,i)=\frac{2^{k+1}}{(k+1)(k+2)}-\frac{2^{k+1}}{k+2}$. Therefore, $\sum_{i=1}^k F(k+1,i)-\sum_{i=1}^k F(k,i)=\frac{2^{k+1}}{(k+1)(k+2)}-\frac{2^{k+1}}{k+2}$; equivalently, one term on the left,

$$\sum_{i=1}^{\boldsymbol{k+1}} F(k+1,i)-\sum_{i=1}^{k} F(k,i) = \frac{\boldsymbol{2^{k+2}}}{\boldsymbol{k+2}}+\frac{2^{k+1}}{(k+1)(k+2)}-\frac{2^{k+1}}{k+2}=\frac{2^{k+1}}{k+1}.$$

Iterate this to find $\sum_{i=1}^k \frac{2^{k+1}}{(k+1)\binom{k}{i}}=\sum_{i=1}^k\frac{2^i}{i}$ or that $\sum_{i=1}^k\binom{k}{i}^{-1}=\frac{k+1}{2^{k+1}}\sum_{i=1}^k\frac{2^i}{i}$. Consequently,

$$\sum_{1\le i\le k\le n}\frac{(-2)^k\binom{n}{k}}{(k+1)\binom{k}{i}}=\sum_{k=1}^n\frac{(-2)^k\binom{n}{k}}{k+1}\sum_{i=1}^n\frac{1}{\binom{k}{i}}=\frac12\sum_{k=1}^n(-1)^k\binom{n}{k}\sum_{i=1}^k\frac{2^i}{i}:=f(n).$$

Using Pascal's recurrence $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ one obtains

$$\begin{aligned} f(n)&=\frac12\sum_{k=1}^n(-1)^k\binom{n-1}{k}\sum_{i=1}^k\frac{2^i}{i}+\frac12\sum_{k=1}^n(-1)^k\binom{n-1}{k-1}\sum_{i=1}^k\frac{2^i}{i}\\ &=\frac12\sum_{k=1}^{\boldsymbol{n-1}}(-1)^k\binom{n-1}{k}\sum_{i=1}^k\frac{2^i}{i}-\frac12\sum_{\ell=0}^{n-1}(-1)^\ell\binom{n-1}{\ell}\sum_{i=1}^{\ell+1}\frac{2^i}{i}\\ &=-\sum_{\ell=0}^{n-1}(-1)^\ell\binom{n-1}{\ell}\cdot\frac{2^\ell}{\ell+1}=-\sum_{\ell=0}^{n-1}\frac{(-2)^\ell}{\ell+1}\cdot\binom{n-1}{\ell}. \end{aligned}$$

Since $\sum_{\ell=0}^{n-1}\binom{n-1}{\ell}(-x)^\ell=(1-x)^{n-1}$, we get $2\sum_{\ell=0}^{n-1}\frac{(-2)^\ell}{\ell+1}\binom{n-1}{\ell}=\int_0^2(1-x)^{n-1}dx=\frac{1-(-1)^n}{n}$. In other words, $f(n)=\frac{(-1)^n-1}{2n}$ as desired. □

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