SOLUTION TO PROBLEM #12396

Problem #12396. Proposed by H. Ohtsuka (japan). If F_n is the n-th Fibonacci number, then prove

$$\sum_{n=1}^{\infty} \arctan\left(\frac{1}{3F_n^2}\right) = \frac{\pi}{4}.$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Define the sequence $x_n = F_{n-2}F_{n-1} + F_nF_{n+1}$ which equals $x_n = 2F_n^2 - F_{n-2}^2$ since $x_n = F_{n-1}F_{n-2} + F_n(F_n + F_{n-1}) = F_n^2 + F_{n-1}(F_n + F_{n-2}) = F_n^2 + (F_n - F_{n-1})(F_n + F_{n-2}) = F_n^2 + F_n^2 - F_{n-2}^2$. As a first step, we show

$$1 + x_n x_{n+1} = 3F_n^2(x_{n+1} - x_n). \tag{1}$$

Because $F_1^2 + \cdots + F_n^2 = F_nF_{n+1}$, we get $x_{n+1} - x_n = F_{n+1}^2 + F_{n-1}^2$. Thus, (1) amounts to $1 + (2F_n^2 - F_{n-2}^2)(2F_{n+1}^2 - F_{n-1}^2) = 3F_n^2F_{n+1}^2 + 3F_n^2F_{n-1}^2$. Routine expansion and re-arrangement leads to $1 - F_{n-2}^2F_{n+1}^2 - 4F_n^2F_{n-1}^2 = (F_{n-1}^2 - F_{n+1}^2)(F_n^2 - F_{n-2}^2)$. Using $F_{n-2}F_{n+1} = F_{n-1}F_n - (-1)^n$,

$$1 - F_{n-2}^2F_{n+1}^2 - 4F_{n-1}^2F_n^2 = 1 - [F_{n-1}F_n - (-1)^n]^2 - 4F_{n-1}^2F_n^2 = -F_{n-1}F_n[5F_{n-1}F_n - 2(-1)^n].$$

On the other hand, $(F_{n-1}^2 - F_{n+1}^2)(F_n^2 - F_{n-2}^2) = -F_{n-1}F_n(F_{n-1} + F_{n+1})(F_n + F_{n-2})$. After canceling out $F_{n-1}F_n$, the task reduces to $5F_{n-1}F_n - 2(-1)^n = (F_{n-1} + F_{n+1})(F_n + F_{n-2})$. From $F_{n-2}F_{n+1} = F_{n-1}F_n - (-1)^n$, we gather $(F_{n-1}+F_{n+1})(F_n+F_{n-2}) = F_{n-1}F_n + F_nF_{n+1} + F_{n-1}F_{n-2} + F_{n-2}F_{n+1} = 2F_{n-1}F_n + F_nF_{n+1} + F_{n-1}F_{n-2} - (-1)^n$. Hence, $5F_{n-1}F_n - 2(-1)^n = 2F_{n-1}F_n + F_nF_{n+1} + F_{n-1}F_{n-2} - (-1)^n$ which says $3F_{n-1}F_n - F_nF_{n+1} - F_{n-1}F_{n-2} = (-1)^n$. The LHS becomes, $F_{n-1}F_n + (F_{n-1}F_n - F_{n-1}F_{n-2}) + (F_{n-1}F_n - F_nF_{n+1}) = F_{n-1}F_n + F_{n-1}^2 - F_n^2 = F_{n-1}F_{n+1} - F_n^2$. We arrive at $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, which is *Cassini's identity.* So, (1) follows.

Due to (1), the sum turns into a telescoping series, and hence

$$\begin{aligned}\sum_{n=1}^{\infty} \arctan\left(\frac{1}{3F_n^2}\right) &= \sum_{n=1}^{\infty} \arctan\left(\frac{x_{n+1} - x_n}{1 + x_nx_{n+1}}\right) = \sum_{n=1}^{\infty} [\arctan(x_{n+1}) - \arctan(x_n)] \\ &= \lim_{N\to\infty} \arctan(x_{N+1}) - \arctan(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \qquad \square\end{aligned}$$