## SOLUTION TO PROBLEM #12398

*Problem #12398. Proposed by L. Glasser (USA).* Evaluate

$$\sum_{n=0}^{\infty} \frac{1}{\sinh(2^n)}.$$

*Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA.* Just use the definition $\frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}} = \frac{2e^{-x}}{1-e^{-2x}}$ and the geometric series $\frac{1}{1-y} = \sum_{k\geq 0} y^k$ to proceed in the manner

$$\sum_{n\geq 0} \frac{1}{\sinh(2^n)} = 2\sum_{n\geq 0} \frac{e^{-2^n}}{1-e^{2\cdot 2^n}} = 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} e^{-2^n} e^{-2k\cdot 2^n} = 2\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} e^{-(2k+1)\cdot 2^n}.$$

Since each positive integer $j$ can be *uniquely* expressed as $j = (2k+1)2^n$, we may convert the double sum into a single sum and apply the formula $\sum_{j=1}^{\infty} x^j = \frac{x}{1-x}$ to get

$$\sum_{n\geq 0} \frac{1}{\sinh(2^n)} = 2\sum_{j=1}^{\infty} e^{-j} = \frac{2e^{-1}}{1-e^{-1}} = \frac{2}{e-1} \qquad \square$$

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