SOLUTION TO PROBLEM #12407

Problem #12407. Proposed by Anonymous (India). Let r be a positive real number. Evaluate

$$\int_0^\infty \frac{x^{r-1}}{(1+x^2)(1+x^{2r})}\,dx.$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Denote the integral by I and make the substitution $y=\frac{1}{x}$ so that

$$I=\int_0^\infty \frac{y^{r+1}}{(1+y^2)(1+y^{2r})}\,dy.$$

Averaging out the two forms of the integral gives $I=\frac{1}{2}\int_0^\infty \frac{z^{r-1}}{1+z^{2r}}\,dz$. Again, change variables with $w=z^r$ so that $I=\frac{1}{2r}\int_0^\infty \frac{dw}{1+w^2}$. Using the well-known (and elementary) fact $\int_0^\infty \frac{dw}{1+w^2}=\frac{\pi}{2}$, one obtains $I=\frac{1}{2r}\frac{\pi}{2}=\frac{\pi}{4r}$. $\square$

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