SOLUTION TO PROBLEM #12412

Problem #12412. Proposed by R. Stanley, University of Miami, Coral Gables, FL, USA. For $n \geq 1$, let $f(n) = \sum_d d^{n/d}(n/d)!$, where the sum is over all positive squarefree divisors of n. Prove that $f(n)$ is divisible by n^2.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, and Akalu Tefera, Grand Valley State University, MI, USA. Let $n = p_1^{a_1} \cdots p_k^{a_k}$ for primes $p_i \neq p_j$ when $i \neq j$. Suppose some exponent exceeds 1, w.l.o.g say $a_1 > 1$. We show that $p_1^{2a_1} \mid f(n)$. Our function has the form

$$f(n) = \sum_{s=1}^{k} \sum_{1 \leq i_1 < \cdots < i_s \leq k} (p_{i_1} \cdots p_{i_s})^{\frac{n}{p_{i_1} \cdots p_{i_s}}} \left(\frac{n}{p_{i_1} \cdots p_{i_s}} \right)!.$$

Denote $X_s := (p_{i_1} \cdots p_{i_s})^{\frac{n}{p_{i_1} \cdots p_{i_s}}} \left(\frac{n}{p_{i_1} \cdots p_{i_s}} \right)!$. Then $\nu_{p_1}(X_s) \geq \nu_{p_1}\left(p_1^{p_1^{a_1-1}} (p_1^{a_1-1})! \right) = \frac{p_1^{a_1}-1}{p_1-1}$ when $p_1 \in \{p_{i_1}, \dots, p_{i_s}\}$; otherwise, if $p_1 \notin \{p_{i_1}, \dots, p_{i_s}\}$ then $\nu_{p_1}(X_s) \geq \nu_{p_1}((p_1^{a_1})!) = \frac{p_1^{a_1}-1}{p_1-1}$. We used Legendre's formula $\nu_p(m!) = \frac{m - s_p(m)}{p-1}$ where $s_p(m)$ equals the sum of the p-adic digits of m. If $p_1 \geq 3$ then $\frac{p_1^{a_1}-1}{p-1} \geq 2a_1$ which results in $p_1^{2a_1} \mid f(n)$. The same is true for $p_1 = 2$ and $a_1 \geq 3$. It is routine to verify this when $p_1 = 2$ and $a_1 = 2$, separately.

We may now safely assume that $n = p_1 \cdots p_k$, a product of distinct primes. First specialize to the case $n = pq$. Thus $d \in \{p, q, pq\}$ and hence $f(n) = p^q q! + q^p p! + pq = pq(p^{q-1}(q-1)! + q^{p-1}(p-1)! + 1)$. Wilson's Theorem shows $(p-1)! \equiv -1 \pmod p$ while Fermat's Little Theorem gives the congruence $q^{p-1} \equiv 1 \pmod p$. Since $q - 1 \geq 1$, we know $p^{q-1} \equiv 0 \pmod p$. Combining these facts, one obtains

$$p^{q-1}(q-1)! + q^{p-1}(p-1)! + 1 \equiv 0 + q^{p-1}(p-1)! + 1 \equiv 0 + (1)(-1) + 1 = 0 \pmod p.$$

Similarly q divdes $p^{q-1}(q-1)! + q^{p-1}(p-1)! + 1$, and therefore by the Chinese Remainder Theorem $n^2 = p^2q^2 \mid f(n)$ because $\gcd(p, q) = 1$.

Generally, let $n = p_1 \cdots p_k$ for distinct primes. In the present case, we may write (slightly differently)

$$f(n) = \sum_{s=1}^{k} \sum_{1 \leq i_1 < \cdots < i_s \leq k} \left(\frac{n}{p_{i_1} \cdots p_{i_s}} \right)^{p_{i_1} \cdots p_{i_s}} (p_{i_1} \cdots p_{i_s})!.$$

Let $s \geq 2$ and fix $1 \leq j \leq k$. If $p_j \in \{p_{i_1}, \dots, p_{i_s}\}$ then $p_j^2 \mid (2p_j)! \mid (p_{i_1} \cdots p_{i_s})!$. If $p_j \notin \{p_{i_1}, \dots, p_{i_s}\}$ then $p_j^2 \mid \left(\frac{n}{p_{i_1} \cdots p_{i_s}} \right)^{p_{i_1} \cdots p_{i_s}}$. So, when $s \geq 2$ we get $n^2 = p_1^2 \cdots p_k^2 \mid \left(\frac{n}{p_{i_1} \cdots p_{i_s}} \right)^{p_{i_1} \cdots p_{i_s}} (p_{i_1} \cdots p_{i_s})!$. That leaves only to study $p_1 \cdots p_k + \sum_{i=1}^{k} \left(\frac{n}{p_i} \right)^{p_i} (p_i)! = p_1 \cdots p_k \left[1 + \sum_{i=1}^{k} \left(\frac{n}{p_i} \right)^{p_i - 1} (p_i - 1)! \right]$. The argument for n^2 divides the latter is *mutatis mutandis* to the case $n = pq$ we considered above. Therefore, we arrive at $n^2 \mid f(n)$ under the hypothesis $n = p_1 \cdots p_k$. The proof is complete. □

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