SOLUTION TO PROBLEM #12413

Problem #12413. Proposed by Seewoo Lee, Berkeley, CA, USA. For a positive real number r, let $I_r = \int_0^{\frac{\pi}{2}} \sin^r \theta \, d\theta$. Prove

$$\frac{1}{(r+1)^2} + I_{r+1}^2 < \left(\frac{r+3}{r+2}\right)^2 I_r^2$$

for all $r \geq 1$.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, and Akalu Tefera, Grand Valley State University, MI, USA. It is known that $I_r = \binom{r/2}{1/2}^{-1} = \frac{\Gamma(\frac{r}{2}+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{r}{2}+1)}$ where $\Gamma(x)$ stands for the Euler's gamma function. That means, our task reads as:

$$\frac{1}{(r+1)^2} + \frac{\Gamma^2(\frac{r}{2}+1)\Gamma^2(\frac{3}{2})}{(\frac{r}{2}+\frac{1}{2})^2\Gamma^2(\frac{r}{2}+\frac{1}{2})} < \left(\frac{r+3}{r+2}\right)^2 \frac{\Gamma^2(\frac{r}{2}+\frac{1}{2})\Gamma^2(\frac{3}{2})}{\Gamma^2(\frac{r}{2}+1)}.$$

Using $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$ and rearranging terms, here is an equivalent formulation

$$\frac{4}{\pi}(r+2)^2 \cdot \frac{\Gamma^2(\frac{r}{2}+1)}{\Gamma^2(\frac{r}{2}+\frac{1}{2})} + 4\,(r+2)^2 \cdot \frac{\Gamma^4(\frac{r}{2}+1)}{\Gamma^4(\frac{r}{2}+\frac{1}{2})} < (r+1)^2(r+3)^2. \tag{1}$$

The gamma function is log-convex, and in fact (the Bohr-Mollerup theorem) it is the only log-convex function satisfying its recursion $\Gamma(x+1) = x\Gamma(x)$ and the initial condition $\Gamma(1) = 1$. Define the function $f(y) := \log\Gamma(x+y) - y\log x$. Direct calculation shows that $f''(y) \geq 0$ (i.e. f is convex) for $y > -x$. In addition, $f(1) = \log\Gamma(x+1) - \log x = \log\Gamma(x) = f(0)$ and hence $f(y) \leq \log\Gamma(x)$ provided $0 \leq y \leq 1$. That is, $\frac{\Gamma(x+y)}{\Gamma(x)} \leq x^y$. Apply this result to (1) with $x = \frac{r}{2}+\frac{1}{2}$ and $y = \frac{1}{2}$:

$$\begin{aligned}(r+2)^2\left[\frac{4}{\pi}\cdot\frac{\Gamma^2(\frac{r}{2}+1)}{\Gamma^2(\frac{r}{2}+\frac{1}{2})} + 4\cdot\frac{\Gamma^4(\frac{r}{2}+1)}{\Gamma^4(\frac{r}{2}+\frac{1}{2})}\right] &\leq (r+2)^2\left[\frac{4}{\pi}\left(\frac{r}{2}+\frac{1}{2}\right) + 4\left(\frac{r}{2}+\frac{1}{2}\right)^2\right] \\ &= (r+1)(r+2)^2\left[\frac{2}{\pi} + (r+1)\right] < (r+1)(r+2)^3,\end{aligned}$$

where the last inequality is due to $\frac{2}{\pi} < 1$. To prove (1), it suffices to test $(r+1)(r+2)^3 < (r+1)^2(r+3)^2$ which, in turn, tantamount $0 < r^2+3r+1$ after some reductions. The proof follows. □

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