SOLUTION TO PROBLEM #12415

Problem #12415. Proposed by R. Tauraso (Italy). For any non-negative integer n, evaluate

$$\sum_{j=0}^{2n}\sum_{k=\lceil \frac{j}{2}\rceil}^{j}\binom{2n+2}{2k+1}\binom{n+1}{2k-j}.$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA and Akalu Tefera, Grand Valley State University, MI, USA. We interchange the summation order and intend to prove our claim that $S_{n+1}:=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n+2}{2k+1}\binom{n+1}{j}=2^{3n+1}$. From the all-familiar Pascal's recurrence, we derive $\binom{2n+2}{2k+1}=\binom{2n}{2k+1}+2\binom{2n}{2k}+\binom{2n}{2k-1}$ and $\binom{n+1}{j}=\binom{n}{j}+\binom{n}{j-1}$. Thus,

$$\begin{aligned}
S_{n+1}&=\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k+1}\binom{n}{j}+\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k-1}\binom{n}{j-1}\\
&+2\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k}\binom{n+1}{j}+\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k+1}\binom{n}{j-1}+\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k-1}\binom{n}{j}\\
&=2S_n+2\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k}\binom{n+1}{j}+\sum_{k=0}^{n}\sum_{j=1}^{k}\binom{2n}{2k+1}\binom{n}{j-1}+\sum_{k=1}^{n}\sum_{j=0}^{k}\binom{2n}{2k-1}\binom{n}{j}\\
&=2S_n+2\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k}\binom{n+1}{j}+\sum_{k=0}^{n}\sum_{j=1}^{k}\binom{2n}{2k+1}\binom{n}{j-1}+S_n+\sum_{k=0}^{n-1}\binom{2n}{2k+1}\binom{n}{k+1}\\
&=2S_n+2\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k}\binom{n+1}{j}+S_n-\sum_{k=0}^{n-1}\binom{2n}{2k+1}\binom{n}{k}+S_n+\sum_{k=0}^{n-1}\binom{2n}{2k+1}\binom{n}{k+1}.
\end{aligned}$$

Let $[x^a]F(x)$ denote the coefficient of x^a in series for $F(x)$. The quantities $\sum_{k=0}^{n-1}\binom{2n}{2k+1}\binom{n}{k+1}$ and $\sum_{k=0}^{n-1}\binom{2n}{2k+1}\binom{n}{k}$ are the coefficients $[x^{2n+1}]P(x)$ and $[x^{2n-1}]P(x)$ where $P(x):=(1+x^2)^n(1+x)^{2n}$, respectively. However, $P(x)$ is palindromic and so the two sums are equal. Thus, we have $S_{n+1}=4S_n+2\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k}\binom{n+1}{j}$. Next, we show that $\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{2n}{2k}\binom{n+1}{j}=2S_n$ which is equivalent to: $[x^{2n}]\frac{(1+x)^{2n}(1+x^2)^{n+1}}{1-x^2}=[x^{2n-1}]\frac{2(1+x)^{2n}(1+x^2)^n}{1-x^2}$. Clearly, this amount to saying that the coefficient of x^{2n} is zero in the difference

$$\frac{(1+x)^{2n}(1+x^2)^{n+1}}{1-x^2}-\frac{2x(1+x)^{2n}(1+x^2)^n}{1-x^2}=(1-x)(1+x)^{2n-1}(1+x^2)^n.$$

Since $[x^{2n}](1-x)(1+x)^{2n-1}(1+x^2)^n=[x^{2n}](1-x^2)(1+2x+x^2)^{n-1}(1+x^2)^n$. The latter is equal to $[x^{2n}](1-x^2)(1+x^2)^{2n-1}=[x^{2n}](1+x^2)^{2n-1}-[x^{2n-2}](1+x^2)^{2n-1}=\binom{2n-1}{n}-\binom{2n-1}{n-1}=0$, as projected. We gather $S_{n+1}=4S_n+4S_n=8S_n$. We check that $S_1=2$ and conclude that $S_n=2^{3n-2}$. □

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