SOLUTION TO PROBLEM #12421

Problem #12421. Proposed by I. Casu, Romania. Let S be a finite set of real numbers, and let T be the set of all n-by-n matrices having entries in S. Prove

$$\sum_{A\in T}\operatorname{trace}(A^2)=\sum_{A\in T}(\operatorname{trace}(A))^2.$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Using the very definition of trace, we proceed in the manner

$$\begin{aligned}
\sum_{A\in T}\operatorname{trace}(A^2) &= \sum_{A\in T}\sum_{i=1}^n [A^2]_{ii} = \sum_{A\in T}\sum_{i=1}^n\sum_{j=1}^n A_{ij}A_{ji} \\
&= \sum_{i=1}^n\sum_{j=1}^n\left[\sum_{A\in T} A_{ji}A_{ij}\right] = \sum_{i=1}^n\sum_{j=1}^n\left[\sum_{x\in S, y\in S}\ \sum_{\substack{A\in T\\ A_{ij}=x, A_{ji}=y}} xy\right] \\
&= \sum_{i=1}^n\sum_{j=1}^n\left[\sum_{x\in S, y\in S} xy\cdot \#\{A\in T: A_{ij}=x, A_{ji}=y\}\right] \\
&= \sum_{i=1}^n\sum_{j=1}^n\left[\sum_{x\in S, y\in S} xy\cdot \#\{A\in T: A_{ii}=x, A_{jj}=y\}\right] \\
&= \sum_{i=1}^n\sum_{j=1}^n\left[\sum_{x\in S, y\in S}\ \sum_{\substack{A\in T\\ A_{ii}=x, A_{jj}=y}} xy\right] = \sum_{i=1}^n\sum_{j=1}^n\left[\sum_{A\in T} A_{ii}A_{jj}\right] \\
&= \sum_{A\in T}\left[\sum_{i=1}^n A_{ii}\right]\left[\sum_{j=1}^n A_{jj}\right] = \sum_{A\in T}\left[\sum_{i=1}^n A_{ii}\right]^2 = \sum_{A\in T}(\operatorname{trace}(A))^2. \qquad \square
\end{aligned}$$

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