SOLUTION TO PROBLEM #12440

Problem #12440. Proposed by H. Ohtsuka (Japan). Let n be a positive integer. Prove

$$\sum_{k=0}^{n-1} \frac{1}{k+1}\binom{2k}{k} = 2\sum_{k=1}^{n}\binom{2n}{n-k}\sin\left(\frac{(4k+1)\pi}{6}\right).$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Denote the left and right-hand sides of the above equation by f_n and g_n, respectively. If $C_k := \frac{\binom{2k}{k}}{k+1}$ then $f_{n+1}-f_n = C_n$. Clearly $2\sin(\frac{(4k+1)\pi}{6}) = [1,-2,1]$, periodically, for $k \geq 1$. We may thus rewrite

$$g_n = \sum_{k=1}^{n}\binom{2n}{n-k} - 3\sum_{k\geq 0}\binom{2n}{n-3k-2}.$$

Since $4^n = \sum_{k=-n}^{n}\binom{2n}{n-k} = 2\sum_{k=1}^{n}\binom{2n}{n-k} + \binom{2n}{n}$, we obtain $\sum_{k=1}^{n}\binom{2n}{n-k} = 2^{2n-1} - \binom{2n-1}{n}$. From Pascal's recurrence, $\binom{2n+2}{n-3k-1} = \binom{2n}{n-3k-1} + 2\binom{2n}{n-3k-2} + \binom{2n}{n-3k-3}$. We combine these to get

$$\begin{aligned}
g_{n+1}-g_n &= \sum_{k=1}^{n+1}\binom{2n+2}{n+1-k} - \sum_{k=1}^{n}\binom{2n}{n-k} \\
&\quad + 3\sum_{k\geq 0}\binom{2n}{n-3k-2} - 3\sum_{k\geq 0}\left[\binom{2n}{n-3k-1} + 2\binom{2n}{n-3k-2} + \binom{2n}{n-3k-3}\right] \\
&= \sum_{k=1}^{n+1}\binom{2n+2}{n+1-k} - \sum_{k=1}^{n}\binom{2n}{n-k} \\
&\quad - 3\sum_{k\geq 0}\left[\binom{2n}{n-3k-1} + \binom{2n}{n-3k-2} + \binom{2n}{n-3k-3}\right] \\
&= \sum_{k=1}^{n+1}\binom{2n+2}{n+1-k} - \sum_{k=1}^{n}\binom{2n}{n-k} - 3\sum_{k=1}^{n}\binom{2n}{n-k} \\
&= 2^{2n+1} - \binom{2n+1}{n+1} - 4(2^{2n-1}) + 4\binom{2n-1}{n} = C_n.
\end{aligned}$$

That means $f_{n+1}-f_n = g_{n+1}-g_n = C_n$. Also, $f_1 = g_1 = 1$. Obviously, $f_n = g_n$ as desired. □