SOLUTION TO PROBLEM #12449

Problem #12449. Proposed by Veselin Jungić, Simon Fraser University, Burnaby, Canada. Let n be a positive integer with $n \geq 2$. The squares of an (n^2+n-1)-by-(n^2+n-1) grid are colored with up to n colors. Prove that there exist two rows and two columns whose four squares of intersection have the same color.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Let $\frac{N}{n} \geq 2$, and let

$$m := \left\lceil \frac{\binom{N}{2}}{\binom{\frac{N}{n}}{2}} \right\rceil + 1.$$

Claim. *Let $X_1, X_2, \dots, X_N$ be different points. If we have m many different n-colorings of $X_1, \dots, X_N$, then there are two points and two colorings such that both points in both of the colorings have the same color.*

Proof. For a fixed coloring of $X_1, \dots, X_N$, we say two points are said to form a "couple" if they get the same color. First we give a lower bound on the number of "couples" in the above coloring. Denote by c_i the number of points of the i^{th} color. Obviously, $c_1 + \cdots + c_n = N$. So we have $\binom{c_1}{2} + \cdots + \binom{c_n}{2}$ "couples". By Jensen's inequality,

$$\binom{c_1}{2} + \cdots + \binom{c_n}{2} \geq \binom{\frac{c_1+\cdots+c_n}{n}}{2} n = \binom{\frac{N}{n}}{2} n.$$

But there are $\binom{N}{2}$ possible "couples" altogether, and a "couple" can be colored with n different colors. Thus, as long as

$$m\binom{\frac{N}{n}}{2} n > \binom{N}{2} n$$

holds, among any m colorings of $X_1, \dots, X_N$ with n colors we can always find two sharing a common "couple" which receives the same color in both colorings. $\square$

We apply the claim as follows: choose $N = n^2 + n - 1$ and $t = n + 1$. By direct calculation, $m = n^2 + n - 1$. In the given problem, $X_1, \dots, X_{n^2+n-1}$ are the squares in the first row that are covered with n-colors and this being repeated $m = n^2 + n - 1$ times to line up one-after-another and form the desired square grid. The conclusion of the proven claim implies the solution of the problem. $\square$