SOLUTION TO PROBLEM #12457

Problem #12457. Proposed by Hideyuki Ohtsuka, Saitama, Japan. Let F_n be the n-th Fibonacci number, and let $S = \sum_{n=1}^{\infty} \frac{1}{F_n^2}$.

(a) Express $\sum_{n=1}^{\infty} \frac{1}{(F_n F_{n+1} F_{n+2})^2}$ in terms of S.

(b) Express $\sum_{n=1}^{\infty} \frac{1}{(F_n F_{n+2})^3}$ in terms of S.

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. Rearrange *Cassini's identity* $F_{n+1}^2 - F_{n+2}F_n = (-1)^n$, based on $F_{n+1} = F_{n+2} - F_n$, into the form $(-1)^n = F_{n+2}F_{n+1} - F_{n+1}F_n - F_{n+2}F_n$. Consequently, $\frac{(-1)^n}{F_n F_{n+1} F_{n+2}} = \frac{1}{F_n} - \frac{1}{F_{n+2}} - \frac{1}{F_{n+1}}$. Squaring both sides and using $F_{n+2} - F_n = F_{n+1}$, again, leads to

$$\begin{aligned}
\frac{1}{(F_n F_{n+1} F_{n+2})^2} &= \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} - \frac{2}{F_n F_{n+2}} - 2\left(\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1}F_{n+2}}\right) \\
&= \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} - \frac{4}{F_n F_{n+2}}.
\end{aligned}$$

Summing over all positive integers (yet again using $F_{n+2} - F_n = F_{n+1}$) results in

$$\begin{aligned}
\sum_{n\geq 1} \frac{1}{(F_n F_{n+1} F_{n+2})^2} &= \sum_{n\geq 1} \left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right) - \sum_{n\geq 1} \frac{4}{F_n F_{n+2}} \\
&= \left(3S - \frac{1}{F_1^2} - \frac{1}{F_1^2} - \frac{1}{F_2^2}\right) - 4\sum_{n\geq 1} \left(\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1}F_{n+2}}\right) \\
&= (3S - 3) - 4\left(\frac{1}{F_1 F_2}\right) = 3S - 7.
\end{aligned}$$

The answer to part (a) is $3S - 7$. We now turn our attention to the second sum (b). Proceed with

$$\begin{aligned}
\frac{1}{(F_n F_{n+2})^3} &= \left(\frac{1}{F_n F_{n+1}} - \frac{1}{F_{n+1}F_{n+2}}\right)^3 \\
&= \frac{1}{(F_n F_{n+1})^3} - \frac{1}{(F_{n+1}F_{n+2})^3} - \frac{3}{F_n F_{n+1}^3 F_{n+2}} \left(\frac{1}{F_n} - \frac{1}{F_{n+2}}\right).
\end{aligned}$$

We may now sum over all positive integers and apply part (a) such that

$$\begin{aligned}
\sum_{n\geq 1} \frac{1}{(F_n F_{n+2})^3} &= \sum_{n\geq 1} \left(\frac{1}{(F_n F_{n+1})^3} - \frac{1}{(F_{n+1}F_{n+2})^3}\right) - 3\sum_{n\geq 1} \frac{1}{(F_n F_{n+1} F_{n+2})^2} \\
&= \frac{1}{(F_1 F_2)^3} - 3\cdot(3S - 7) = 22 - 9S.
\end{aligned}$$

The answer to part (b) is $22 - 9S$. The argument is complete. □

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