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Problem #1540: [P]
(a) Show that $x^n + (x-1)^n - (x+1)^n$ has a unique non-zero real root r_n.
(b) Show that r_n increases monotonically.
(c) Evaluate $\lim_{n\to\infty} r_n/n$.

Solution: Let $f_n(x) := x^n + (x-1)^n - (x+1)^n$. Notice that if $x = -t < 0$, then we have

$$f_n(-t) = (-1)^n[t^n + (t+1)^n - (t-1)^n]$$

is either always positive or negative (depending on n). Thus it suffices to look for the positive roots of the f_n's.

Claim: $\exists!\ r_n > 0$ such that $f_n(x) < 0$ for $0 < x < r_n$; $f_n(x) > 0$ for $x > r_n$; $f_n(r_n) = 0$ and $r_n > r_{n-1}$.

We proceed by induction. For $n = 1$, the claim is trivial: $f_1(x) = x - 2$, and $r_1 = 2$. Assume the claim holds for $n-1$. Then, since

$$f_n(x) = (-1)^n - 1 + n\int_0^x f_{n-1}(t)dt,$$

we have $f_n(x) \le 0$ for $0 < x \le r_{n-1}$. But $f_n(\infty) = \infty$. So, by the Intermediate Value Theorem there is $r_n > r_{n-1} > 0$ a root of $f_n(x)$. Moreover, $f'_n(x) = nf_{n-1}(x)$ implies that $f_n(x)$ is strictly increasing once $x > r_{n-1}$. Consequently, such a solution is unique and hence the claim follows.

Now, the equations $0 = r_n^n + (r_n - 1)^n - (r_n+1)^n$ may be rewritten as

$$\left(1 + \frac{1}{r_n}\right)^{r_n(n/r_n)} - \left(1 - \frac{1}{r_n}\right)^{r_n(n/r_n)} = 1. \tag{1}$$

Taking limits in the last equation, we obtain

$$e^{\lim_{n\to\infty} n/r_n} - e^{-\lim_{n\to\infty} n/r_n} = 1. \tag{2}$$

With this, n/r_n must be bounded (else equation (2) leads to Cont radication, $\infty = 1$). Hence each subsequence has a convergent subsequence, and for such a convergent subsequence, the common limit, say L, can be determined from $e^L - e^{-L} = 1$. This in turn gives $L = arcsinh(1/2) = ln((1+\sqrt{5})/2)$. Therefore, the desire limit is

$$\lim_{n\to\infty} \frac{r_n}{n} = 1/ln\left(\frac{1+\sqrt{5}}{2}\right).$$

References:

[**P**] P 1540, *Mathematics Magazine,* **(71) #1,** February 1998.