SOLUTION TO PROBLEM #1546
PROPOSED BY B. G. KLEIN, ET AL

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Problem #1546: [**P**] Given $y > 1$, let P be the set of all real polynomials $p(x)$ with nonnegative coefficients that satisfy $p(1) = 1$ and $p(3) = y$. Prove there exists $p_0(x) \in P$ such that

(i) $\{p(2) : p(x) \in P\} = (1, p_0(2)]$;
(ii) if $p(x) \in P$ and $p(2) = p_0(2)$, then $p(x) = p_0(x)$.

Solution: We show that $p_0(x) = \frac{y-1}{2}(x-1) + 1$ is the unique polynomial satisfying the above conditions with $p_0(2) = (y+1)/2$.

Note that since coefficients are nonnegative, *all* functions in P are strictly increasing as well as concave upwards for $1 \le x \le 3$. Thus certainly for each $p(x) \in P$, we have $p(x) \le p_0(x), x \in [1,3]$ since the line $p_0(x)$ joins the end points $(1,1)$ and $(3,y)$.

Furthermore, $p_0(x)$ is unique. Else assume that $p(2) = p_0(2)$, for some $p_0(x) \neq p(x) \in P$. Then by the Intermediate-Value-Theorem for derivatives, there exist two distinct points (one in $(1,2)$ and another in $(2,3)$) where tangents to $p(x)$ (hence derivatives) have same slope, that is, $(y-1)/2$. This cannot be true of the nonlinear $p(x)$ as its derivative is one-to-one (because $p''(x) > 0$ there). Contradiction.

To prove (i), first note that strict monotonicity implies that for $p(x) \in P$, we have $p(2) > 1$. Moreover it is easy to manufacture polynomials in P with $p(2) = \alpha$, for a given $1 < \alpha < (y+1)/2$. For example, one may use $p(x) = ax^n + bx + c$ with suitable constants a, b, c and n. □

References:

[**P**] P 1546, *Mathematics Magazine,* **(71) #2,** April 1998.

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