# SOLUTION TO PROBLEM #1554
# PROPOSED BY H. C. MORRIS

Tewodros Amdeberhan

DeVry Institute, Mathematics
630 US Highway One, North Brunswick, NJ 08902
amdberhan@admin.nj.devry.edu

**Problem #1554:** [**P**] For $0 \leq r \leq 1$, find the volume $V_n(r)$ of

$$\left\{(x_1, ..., x_n) \in [0,1]^n : \prod_{i=1}^{n} x_i \leq r\right\}.$$

**Solution:** Denote the above set by $S_n(r)$, then observe that this set is the cube $I^n := [0,1]^n$ with its top right portion chopped off by the hyperboloid $\{(x_1, ..., x_n) \in R^n : \prod_{i=1}^{n} x_i = r\}$. Hence, for $n > 1$ we have

$$\begin{aligned} V_n(r) = \int_{S_n(r)} dx &= V(I^n) - \int_{[r,1]^{n-1}} \left(1 - \frac{r}{x_1 \cdots x_{n-1}}\right) dx_1 \cdots x_{n-1}. \\ &= V(I^n) - V([r,1]^{n-1}) + r\left(\int_r^1 \frac{1}{y} dy\right)^{n-1} \\ &= 1 - (1-r)^{n-1} + r(-1)^{n-1} ln^{n-1}(r). \end{aligned}$$

Since clearly $V_1(r) = r$, we get $V_n(r) = 1 - (1-r)^{n-1} + r(-1)^{n-1} ln^{n-1}(r)$, *for all* $n \geq 1$. □

**References:**

[**P**] P 1554, *Mathematics Magazine,* **(71) #4,** October 1998.