SOLUTION TO PROBLEM #1562
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Problem #1562: [**P**] Prove that

$$\tan\left(\frac{1}{4}\tan^{-1}4\right) = 2\left(\cos\frac{6\pi}{17} + \cos\frac{10\pi}{17}\right). \tag{1}$$

Proof: Let $z := 2\cos\frac{6\pi}{17} + 2\cos\frac{10\pi}{17}$. Then equation (1) becomes $\tan^{-1}4 = 4\tan^{-1}z$. Now, a successive application of the identity

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right),$$

once with $x = y = z$ and then with $x = y = \frac{2z}{1-z^2}$, results in

$$\tan^{-1}4 = \tan^{-1}\left[\frac{4z(1-z^2)}{(1-z^2)^2-(2z)^2}\right].$$

This in turn may be simplified to take the form

$$z^4 + z^3 - 6z^2 - z + 1 = 0. \tag{2}$$

Hence, it suffice to prove (2). Denoting the primitive $17th$-root of unity (i.e. $w^{17} = 1$) by w_0, the assertion in (2) is equivalent to the claim that

$$z = \textit{Real part of } 2(w_0^6 + w_0^{10}) = w_0^6 + w_0^{10} + w_0^{11} + w_0^7,$$

is a solution of the polynomial equation (2). Direct computation and the identity

$$w_0^{16} + w_0^{15} + \cdots + w_0 + 1 = 0$$

coupled with $w_0^{17} = 1$, easily verify the latter statement. □

References:

[**P**] P 1562, *Mathematics Magazine,* **(71) #5,** December 1998.