SOLUTION TO PROBLEM #1572 PROPOSED BY WESTERN MARYLAND COLLEGE PROBLEMS GROUP

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Problem #1572: [P] Let $b_0 = 1$ and b_1 satisfy $0 < b_1 < 1$. For $n \geq 1$, define b_{n+1} by

$$b_{n+1} = \frac{2b_n b_{n-1} - b_n^2}{3b_{n-1} - 2b_n}.$$

Show that $(b_n)_{\{n\geq 0\}}$ converges, and compute its limit in terms of b_1.

Solution: From the definition of b_{n+1} and since $b_0 = 1$, it follows that

$$b_{n+1} = b_1 \frac{2-b_1}{3-2b_1} \frac{4-3b_1}{5-4b_1} \cdots \frac{2n-(2n-1)b_1}{2n+1-2nb_1} = b_1 \prod_{k=1}^{n} \frac{2k-(2k-1)b_1}{2k+1-2kb_1},$$

which is easily proven by induction on n. Letting $\beta := 1/(2-2b_1) > 1/2$, and using binomial coefficients we may rewrite this as follows:

$$b_{n+1} = b_1 \prod_{k=1}^{n} \left(1 - \frac{1}{2k+2\beta}\right) = \frac{2\beta - 1}{2\beta\binom{2\beta}{\beta}} \frac{\binom{2n+2\beta}{n+\beta}}{4^n}.$$

Using Stirling's approximation for $n!$ as $n^n e^{-n}\sqrt{2\pi n}$, we get for any $\beta > 1/2$

$$\lim_{n\to\infty} b_{n+1} = (\text{constant}) \cdot \lim_{n\to\infty} \frac{1}{\sqrt{n+\beta}} = 0. \square$$

References:

[**P**] P 1572, *Mathematics Magazine,* **(72) #2,** April 1999.

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