SOLUTION TO PROBLEM #1625 PROPOSED BY MIHÀLY BENCZE, ROMANIA.

Beiment Teclezghi and Tewodros Amdeberhan

BTeclezghi@NJCU.edu, amdberha@admin.nj.devry.edu

Problem #1625: [P] Let $x_1, x_2, \ldots, x_n$ be positive real numbers and let $a_1, a_2, \ldots, a_n$ be positive integers. Prove that

$$\prod_{k=1}^{n}(1+x_k^{1/a_k})^{a_k} \geq \left(1+\left(\prod_{k=1}^{n} x_k\right)^{1/\sum_1^n a_k}\right)^{\sum_1^n a_k}. \tag{1}$$

Solution: *Beimnet Teclezghi, Math. Dep't, New Jersey City Univ., Jersey City, NJ 07305; Tewodros Amdeberhan, DeVry College of Tech., 630 US Highway One, North Brunswick, NJ 08902, USA.* Let $y_k := x_k^{1/\sum_1^n a_k}$, and $p_k := \frac{1}{a_k}\sum_{k=1}^n a_k$. Then $y_k > 0, p_k > 1$ and (1) becomes

$$1 + y_1 \cdots y_n \leq \prod_{k=1}^{n}(1+y_k^{p_k})^{1/p_k}, \qquad \text{where} \qquad \sum_{k=1}^{n}\frac{1}{p_k} = 1. \tag{2}$$

We will proceed to prove (2) by induction on n. For $n = 2$, the corresponding inequality is a direct consequence of Hölder's Sum Inequality. Assume it holds true for n and show for $n+1$.
Suppose $\sum_{k=1}^{n+1}\frac{1}{p_k} = 1$ and let $q_2 := p_{n+1}$, $\frac{1}{q_1}+\frac{1}{q_2} = 1$, and $\alpha_k := \frac{p_k}{q_1} = \frac{p_{n+1}-1}{p_{n+1}}p_k$, for $1 \leq k \leq n$. Note that $\sum_{k=1}^{n}\frac{1}{\alpha_k} = 1$. Applying Hölder's inequality and then following it through with the induction assumption, we obtain

$$\begin{aligned} 1 + y_1 \cdots y_n y_{n+1} &\leq (1+(y_1\cdots y_n)^{q_1})^{1/q_1}\left(1+y_{n+1}^{q_2}\right)^{1/q_2} \\ &\leq \prod_{k=1}^{n}(1+y_k^{q_1\alpha_k})^{\frac{1}{\alpha_k q_1}}\left(1+y_{n+1}^{q_2}\right)^{1/q_2} \\ &= \prod_{k=1}^{n}(1+y_k^{p_k})^{1/p_k}\left(1+y_{n+1}^{p_{n+1}}\right)^{1/p_{n+1}}. \square \end{aligned}$$

References:
[**P**] P 1625, *Mathematics Magazine,* **(74) #3,** June 2001.

NOTE: As the above proof shows, the quantities a_k can be extended to take arbitrary *postive real* values.

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