SOLUTION TO PROBLEM #1626 PROPOSED BY HAJOO LEE, SOUTH KOREA.

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Problem #1626: [**P**] Let $f, g, h : R \to R$ be functions such that $f(g(0)) = g(f(0)) = h(f(0)) = 0$ and

$$f(x + g(y)) = g(h(f(x))) + y$$

for all $x, y \in R$. Prove that $h = f$ and that $g(x + y) = g(x) + g(y)$ for all $x, y \in R$.

Solution: *Beimnet Teclezghi, Math. Dep't, New Jersey City Univ., Jersey City, NJ 07305; Tewodros Amdeberhan, DeVry College of Tech., 630 US Highway One, North Brunswick, NJ 08902, USA.* For $x = y = 0$, we have $0 = f(g(0)) = f(0 + g(0)) = g(h(f(0))) + 0 = g(0)$. Thus $g(0) = 0 = f(0)$. Also,

$$f(x) = f(x + g(0)) = g(h(f(x))) + 0 = g(h(f(x))). \tag{1}$$

Therefore,

$$f(x + g(y)) = f(x) + y. \tag{2}$$

Moreover, $f(g(y)) = f(0 + g(y)) = f(0) + y = y$, that is, for any $y \in R$

$$f(g(y)) = y \qquad \text{and hence } f \text{ is onto } R. \tag{3}$$

Applying f to both sides of equation (1) and using (3), we get $h(f(x)) = f(g(h(x))) = f(f(x))$. Since f is surjective and now $h(f(x)) = f(f(x))$, this implies that $f = h$, and

$$f(x) = g(f^2(x)), \qquad \text{for all } x \in R.$$

Not also that $R =$ Range $f=$ Range $(gf^2) \subset$ Range g. Hence g is onto.

For $a, b \in R$, there exists $c \in R$ such that $g(c) = b$. Then, $f(a + b) = f(a + g(c)) = f(a) + c = f(a) + f(g(c)) = f(a) + f(b)$. Clearly f^2 satisfies the same propery and f^2 is surjective as well.

Now let $x_1, x_2 \in R$. Then there exist $y_1, y_2 \in R$ such that $f^2(y_i) = x_i$ for $i = 1, 2$. If follows that

$$g(x_1) + g(x_2) = g(f^2(y_1)) + g(f^2(y_2)) = f(y_1) + f(y_2) = f(y_1 + y_2) = g(f^2(y_1 + y_2)) = g(x_1 + x_2).$$

This completes the proof of the second-half of the assertion. □

References:

[**P**] P 1626, *Mathematics Magazine,* **(74) #3,** June 2001.

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