SOLUTION TO PROBLEM #623
PROPOSED BY ALEXANDER KHEYFITS

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CLAIM: [**P**] Let k be an integer greater than 1. Then

$$\sum_{n=0}^{\infty}\left[\left(\sum_{j=1}^{k}\frac{1}{kn+j}\right)-\frac{1}{n+1}\right]=\ln(k). \tag{1}$$

PROOF: The partial sum of the series in (1) can be rewritten as:

$$\sum_{n=0}^{m-1}\left[\left(\sum_{j=1}^{k}\frac{1}{kn+j}\right)-\frac{1}{n+1}\right]=\sum_{n=0}^{m-1}\sum_{j=1}^{k}\frac{1}{kn+j}-\sum_{n=0}^{m-1}\frac{1}{n+1}. \tag{2}$$

As j runs over $1,2,\dots,k$ and n runs over $0,1,\dots,m-1$, the quantity $kn+j$ runs over the integers $1,2,\dots,km$, each exactly once! Hence, the sum in equation (2) equals $H_{km}-H_m$, where

$$H_N=\sum_{n=1}^{N}\frac{1}{n}.$$

On the other hand, the Euler's constant γ is given by

$$\gamma=\lim_{m\to\infty}(H_m-\ln(m))=\lim_{m\to\infty}(H_{km}-\ln(km)).$$

This implies that

$$\sum_{n=0}^{\infty}\left[\left(\sum_{j=1}^{k}\frac{1}{kn+j}\right)-\frac{1}{n+1}\right]=\lim_{m\to\infty}(H_{km}-H_m)=\lim_{m\to\infty}(\ln(km)-\ln(m))=\ln(k).$$

This completes the proof. □

References:

[**P**] P #623, *The College Mathematics Journal,* **(29) #2,** 1998.

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