SOLUTION TO PROBLEM #627
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Problem #627: Let x be a positive integer congruent to 1 (mod) 4. If n is any positive integer, show that

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} x^k \qquad \text{is divisible by } 2^{n-1}. \tag{1}$$

Solution: Denote the sum in (1) by $a(n)$. Then first note that it satisfies the recurrence relation

$$a(n+2) - 2a(n+1) - (x-1)a(n) = 0. \tag{2}$$

This is proved *mechanically* using the Maple package $EKHAD$ `(available at http://www.math.temple.edu/zeilberg/programs.html).` *The idea:* let $F(n,k) := \binom{n}{2k+1} x^k$, and $G(n,k) := -\binom{n}{2k-1} x^k$. Then, we have

$$F(n+2,k) - 2F(n+1,k) - (x-1)F(n,k) = G(n,k+1) - G(n,k).$$

Summing over k and telescoping yields the result once equation (2) is checked for $n = 1, 2$ (with a(1)=1, a(2)=2, a(3)=x+3, a(4)=4x+4).

Hence the proof of assertion (1) can be completed by induction.

This is clear for $n = 1$, with $a(1) = 1$. Assume it holds true for *all* integers $\leq n$. In particular, 2^{n-2} divides $a(n-1)$ and 2^{n-1} divides $a(n)$. Now, since $x - 1$ is a multiple of 4 and we have $a(n+1) = 2a(n) + (x-1)a(n-1)$, it follows that $a(n+1)$ is divisible by 2^n. □

References:

[**P**] P #627, *The College Mathematics Journal,* **(29) #3,** 1998.