SOLUTION TO PROBLEM #627
PROPOSED BY GEORGE MACKIW

Tewodros Amdeberhan

DeVry Institute, Mathematics
630 US Highway One, North Brunswick, NJ 08902
amdberhan@admin.nj.devry.edu

Problem #627: Let x be a positive integer congruent to 1 (mod) 4. If n is any positive integer, show that

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} x^k \qquad \text{is divisible by } 2^{n-1}. \tag{1}$$

Solution: Using the binomial expansion of $(x-1+1)^k$ and switching summations, we have

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} x^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} \sum_{j=0}^{k} \binom{k}{j} (x-1)^j = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} (x-1)^j \sum_{k=j}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} \binom{k}{j}.$$

On the other hand,

$$\sum_{k=j}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} \binom{k}{j} = \frac{2^{n-1}}{4^j} \binom{n-j-1}{j}.$$

The identity in the last equation is proved *mechanically* using the Maple package $EKHAD$ `(available at http://www.math.temple.edu/zeilberg/programs.html).`

Consequently,

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} x^k = 2^{n-1} \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \left(\frac{x-1}{4}\right)^j \binom{n-j-1}{j}.$$

Now, since *all* the terms in the second sum are integers, the proof of (1) is complete. □

References:

[**P**] P #627, *The College Mathematics Journal,* **(29) #3,** 1998.

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