SOLUTION TO PROBLEM #643
PROPOSED BY W. F. TRENCH

Tewodros Amdeberhan

DeVry Institute, Mathematics
630 US Highway One, North Brunswick, NJ 08902
amdberhan@admin.nj.devry.edu

Problem #643: Suppose that $a_r \geq 0$ for all nonnegative integers r and $\sum_{r=0}^{\infty} a_r = A < \infty$. Let

$$\alpha_n = \frac{1}{n}\sum_{r=0}^{n-1}\sum_{s=0}^{n-1} a_{r+s} \qquad \text{and} \qquad \beta_n = \frac{1}{n}\sum_{r=0}^{n-1}\sum_{s=0}^{n-1} a_{|r-s|}. \tag{1}$$

Find $\lim_{n\to\infty}\alpha_n$ and $\lim_{n\to\infty}\beta_n$.

Solution: Under the assumption, the power series $f(x) := \sum_{r=0}^{\infty} a_r x^r$ has radius of convergence $R > 1$. Hence, so does its derivative $f'(x) = \sum_{r=1}^{\infty} r a_r x^{r-1}$. In particular, $f'(1) = \sum_{r=1}^{\infty} r a_r$ is finite. Next, we rewrite the sums in (1) respectively as:

$$\alpha_n = \frac{1}{n}\sum_{k=0}^{2n-2}\sum_{r+s=k} a_{r+s} = \frac{1}{n}\left\{\sum_{k=0}^{n-1}(k+1)a_k + \sum_{k=n}^{2n-2}(2n-k-1)a_k\right\} \qquad \text{and}$$

$$\beta_n = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{|r-s|=k} a_{|r-s|} = \frac{1}{n}\sum_{k=1}^{n-1} 2(n-k)a_k + a_0.$$

Consequently, we have

$$\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\frac{1}{n}\left\{\sum_{k=0}^{n-1} 2(k+1)a_k - \sum_{k=n}^{2n-2}(k+1)a_k\right\} + \lim_{n\to\infty} 2\sum_{k=n}^{2n-2} a_k = 0,$$

as the sums are partial sums of convergent series. Similarly,

$$\lim_{n\to\infty}\beta_n = \lim_{n\to\infty}\left\{-a_0 + 2\sum_{k=0}^{n-1} a_k - \frac{2}{n}\sum_{k=1}^{n-1} k a_k\right\} = -a_0 + 2A.$$

Hence we obtain $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\beta_n = 2A - a_0$. $\square$

References:

[**P**] P #643, *The College Mathematics Journal,* **(30) #1,** January 1999.