SOLUTION TO PROBLEM #648
PROPOSED BY AYOUB

[P] Prove that if $z = \sum_{k=0}^{r} \binom{2r+1}{2k+1} 2^k$ where r is a positive integer, then there is a positive integer n such that $n < n+1 < z$ form a Pythagorean triple.

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For each positive integer $x > 0$, consider the positive integers $H_+(x) := \sum_{k=0}^{\infty} \binom{x}{2k} 2^k$ and $H_-(x) := \sum_{k=0}^{\infty} \binom{x}{2k+1} 2^k$, where $\binom{n}{m} = 0$ whenever $m > n$. Then, we prove the

Claim: $z^2 = H_+^2(r+1)H_+^2(r) + 4H_-^2(r+1)H_-^2(r)$ with $|H_+(r+1)H_+(r) - 2H_-(r+1)H_-(r)| = 1$.

Since $H_+(x+1) + \sqrt{2}H_-(x+1) = (\sqrt{2}+1)^{x+1} = (\sqrt{2}+1)(H_+(x) + \sqrt{2}H_-(x))$, we obtain the identities $H_-(x+1) = H_+(x) + H_-(x)$ and $H_+(x+1) = H_+(x) + 2H_-(x)$. Or,

$$(1) \qquad H_+(x) = H_-(x+1) - H_-(x) \qquad \text{and} \qquad H_+(x+1) = H_-(x+1) + H_-(x).$$

Also since

$$H_+(2r+1) + \sqrt{2}H_-(2r+1) = (\sqrt{2}+1)^{2r+1} = [H_+(r+1) + \sqrt{2}H_-(r+1)][H_+(r) + \sqrt{2}H_-(r)],$$

after using (1) it follows that

$$H_-(2r+1) = H_-(r+1)H_+(r) + H_+(r+1)H_-(r) = H_-^2(r+1) + H_-^2(r).$$

Consequently, we have

$$(2) \qquad z^2 = H_-^2(2r+1) = [H_-^2(r+1) + H_-^2(r)]^2 = [H_-^2(r+1) - H_-^2(r)]^2 + 4H_-^2(r+1)H_-^2(r).$$

Rewriting (2) as $z^2 = [H_-(r+1) - H_-(r)]^2[H_-(r+1) + H_-(r)]^2 + 4H_-^2(r+1) + H_-^2(r)$ and combining with (1) results in the first-half of the assertion

$$z^2 = H_+^2(r+1)H_+^2(r) + 4H_-^2(r+1)H_-^2(r).$$

To complete the proof of our claim, observe that

$$\sqrt{2} + 1 = (\sqrt{2}-1)^r(\sqrt{2}+1)^{r+1} = (-1)^r[H_+(r) - \sqrt{2}H_-(r)][H_+(r+1) + \sqrt{2}H_-(r+1)]$$

which shows that $1 = (-1)^r[H_+(r+1)H_+(r) - 2H_-(r+1)H_-(r)]$. □

References:

[**P**] P #648, *The College Mathematics Journal,* **(30) #2,** March 1999.