SOLUTION TO PROBLEM #667
PROPOSED BY LEMMA

[P] Find the value of the integral

$$\int_0^{\pi} \frac{2+2cos(x)-cos((n-1)x)-2cos(nx)-cos((n+1)x)}{1-cos(2x)}dx$$

as a function of n, where n is a nonnegative integer.

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Denote the above integral by f_n, and let $g_n := f_{n+1} - f_n$ and let $h_n := g_n - g_{n-1}$. Then, using the addition trigonometry identities

$cos(A \pm B) = cos(A)cos(B) \mp sin(A)sin(B)$ and $sin(A \pm B) = sin(A)cos(B) \pm cos(A)sin(B)$ we obtain

$$g_n := \int_0^{\pi} \frac{2sin(x)[sin((n+1)x)+sin(nx)]}{1-cos(2x)}dx \qquad \text{and} \qquad h_n := \int_0^{\pi} 2cos(nx)dx.$$

Consequently, this implies the recurrence relations

$$h_n = 0, \qquad g_n = \pi \qquad \text{for} \qquad n \geq 1$$

with initial conditions

$$g_1 = \pi, \qquad f_1 = \pi.$$

Therefore, $f_n = n\pi, n \geq 1$. The case $n = 0$ is obvious. □

References:

[P] P #667, *The College Mathematics Journal,* **(31) #1,** January 2000.

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