SOLUTION TO PROBLEM #705
PROPOSED BY AYOUB

Proposed by Ayoub B. Ayoub, Penn State Univ., Abington College, Abington, PA
If $0 < a \le b$, prove that

$$(*)\qquad a \le \frac{ab(a+b)}{a^2+b^2} \le \sqrt[a+b]{a^b b^a} \le \frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt[a+b]{a^a b^b} \le \frac{a^2+b^2}{a+b} \le b.$$

Solution by Tewodros Amdeberhan, DeVry Institute of Technology.
Observe that the $1^{st}, 2^{nd}, 3^{rd}$, and the 4^{th} inequalities above are reciprocals (multiplied by ab) of the $8^{th}, 7^{th}, 6^{th}$, and the 5^{th} inequalities, respectively. Therefore, it suffices to verify the second half among the chain of estimates in (*). Of these, the last inequality is trivial (use $a \le b$) while $\sqrt{ab} \le \frac{a+b}{2}$ is the well-known AGM-inequality. Thus, we are set to justify only two inequalities. To this end, let $0 < x = a/b \le 1$ and rewrite the inequalities

$$\frac{a+b}{2} \le \sqrt[a+b]{a^a b^b} \le \frac{a^2+b^2}{a+b} \qquad \text{as} \qquad \frac{x+1}{2} \le x^{\frac{x}{x+1}} \le \frac{x^2+1}{x+1}.$$

This is the same as proving

$$\ln(\frac{x+1}{2}) \le \frac{x}{x+1}\ln(x) \le \ln(\frac{x^2+1}{x+1}).$$

The first inequality follows from the convexity ($f''(x) > 0$) of $f(x) = x\ln(x)$ and after rewriting it as

$$\frac{x+1}{2}\ln(\frac{x+1}{2}) \le \frac{x\ln(x)}{2}, \qquad \text{i.e.,} \qquad f(\frac{x+1}{2}) \le \frac{f(x)+f(1)}{2}.$$

Noting that $0 < x, \frac{x}{x+1} \le 1$, the other inequality is immediate

$$\ln(\frac{x^2+1}{x+1}) \ge \ln(\frac{x^2+x}{x+1}) = \ln(x) \ge \frac{x}{x+1}\ln(x).\square$$

References:

[**P**] #705, *The College Mathematics Journal,* **(32) #3,** May 2001.

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