PROOF OF FORMULA 3.197.5

$$\int_0^\infty x^{\lambda-1}(1+x)^\nu(1+ax)^\mu \, dx = B(\lambda, -\mu-\nu-\lambda) \, {}_2F_1\left[-\mu, \lambda; -\mu-\nu; 1-a\right]$$

The change of variables $t = x/(1+x)$ yields

$$\int_0^\infty x^{\lambda-1}(1+x)^\nu(1+ax)^\mu \, dx = \int_0^1 t^{\lambda-1}(1-t)^{-\lambda-\nu-\mu-1}\left[1-(1-a)t\right]^\mu \, dt.$$

The result now comes from the integral representation of the hypergeometric function

$$_2F_1\left[\alpha, \beta; \gamma; z\right] = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-zx)^{-\alpha} \, dx.$$