PROOF OF FORMULA 3.411.19

$$\int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x} = -\sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(p+n-k)$$

Define

$$I(p) := \int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x},$$

and differentiate with repect to p to obtain

$$\begin{aligned} I'(p) &= -\int_0^\infty e^{-px}(e^{-x}-1)^n dx \\ &= -\sum_{k=0}^{n} (-1)^k \binom{n}{k} \int_0^\infty e^{-(p+n-k)x}\, dx \\ &= -\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{p+n-k}. \end{aligned}$$

Therefore

$$I(p) = -\sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(p+n-k) + C.$$

The constant of integration C is determined from the condition $I(+\infty) = 0$. Now observe that

$$\prod_{k=0}^{n} (p+n-k)^{(-1)^k \binom{n}{k}} \sim p^{\sum_{k=0}^{n} (-1)^k \binom{n}{k}} = 1.$$

Therefore, $C = 0$.