PROOF OF FORMULA 3.411.20

$$\int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x^2} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (p+n-k)\ln(p+n-k)$$

Define

$$I(p) = \int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x^2},$$

and differentiate with respect to p to obtain

$$I'(p) = -\int_0^\infty e^{-px}(e^{-x}-1)^n \frac{dx}{x}.$$

It follows from entry **3.411.19** that

$$I'(p) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(p+n-k).$$

Integrate to get

$$I(p) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[(p+n-k)\ln(p+n-k) - (p+n-k) \right] + C.$$

Observe first that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (p+n-k) = (p+n)\sum_{k=0}^{n} (-1)^k \binom{n}{k} - \sum_{k=0}^{n} (-1)^k k \binom{n}{k} = 0,$$

since both sums vanish by considering $(1-x)^n$ and its derivative at $x=1$. Therefore

$$I(p) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (p+n-k)\ln(p+n-k) + C.$$

Now let $p \to \infty$ to see that $C = 0$.