PROOF OF FORMULA 3.452.2

$$\int_0^\infty \frac{x^2\,dx}{\sqrt{e^x-1}} = \frac{\pi}{3}\left(12\ln^2 2+\pi^2\right)$$

The integral is

$$\int_0^\infty \frac{x^2\,dx}{\sqrt{e^x-1}} = \int_0^\infty \frac{x^2 e^{-x/2}\,dx}{\sqrt{1-e^{-x}}},$$

and this can be expressed as $h''(\frac{1}{2})$, where

$$h(a) = \int_0^\infty \frac{e^{-ax}\,dx}{\sqrt{1-e^{-x}}}.$$

To evaluate $h(a)$ let $t = e^{-x}$ to obtain

$$h(a) = \int_0^1 t^{a-1}(1-t)^{-1/2}\,dt = B(a, \tfrac{1}{2}) = \frac{\Gamma(a)\sqrt{\pi}}{\Gamma(a+\frac{1}{2})}.$$

Logarithmic differentiation yields

$$h'(a) = h(a)\left[\psi(a) - \psi(a+\tfrac{1}{2})\right]$$

and

$$h''(a) = h'(a)\left[\psi(a) - \psi(a+\tfrac{1}{2})\right] + h(a)\left[\psi'(a) - \psi'(a+\tfrac{1}{2})\right].$$

The values $\psi(1) = -\gamma$, $\psi(\frac{1}{2}) = -\gamma - 2\ln 2$, $\psi'(\frac{1}{2}) = \pi^2/2$ and $\psi'(1) = \pi^2/6$ yield the result.