PROOF OF FORMULA 3.452.4

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{2x}-1}} = 1 - \ln 2$$

The change of variables $t = 2x$ gives

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{x}-1}} = \frac{1}{4}\int_0^\infty \frac{t\,e^{-t}\,dt}{\sqrt{1-e^{-t}}}.$$

Now define

$$f(a) := \int_0^\infty \frac{e^{-ax}\,dx}{\sqrt{1-e^{-x}}},$$

so that

$$f'(a) = -\int_0^\infty \frac{xe^{-ax}\,dx}{\sqrt{1-e^{-x}}}.$$

Then

$$\int_0^\infty \frac{xe^{-x}\,dx}{\sqrt{e^{2x}-1}} = -\frac{1}{4}f'(1).$$

Now let $u = e^{-x}$ to obtain

$$f(a) = \int_0^1 u^{a-1}(1-u)^{-1/2}\,du = B(a, \tfrac{1}{2}) = \frac{\Gamma(a)\,\Gamma(\frac{1}{2})}{\Gamma(a+\frac{1}{2})}.$$

The integral representation for the beta function

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

was used. Logarithmic differentiation gives

$$f'(a) = f(a)\left[\psi(a) - \psi(a+\tfrac{1}{2})\right].$$

The values $f(1) = 2$ and

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k} \text{ and } \psi(n+\tfrac{1}{2}) = -\gamma + 2\left(\sum_{k=1}^{n}\frac{1}{2k-1} - \ln 2\right),$$

give

$$\psi(1) = -\gamma \text{ and } \psi(\tfrac{3}{2}) = 2 - \gamma - 2\ln 2.$$

Therefore

$$f'(1) = 2\left(-\gamma - (2 - \gamma - 2\ln 2)\right) = -4(1 - \ln 2).$$

This completes the evaluation.