PROOF OF FORMULA 3.462.7

$$\int_0^\infty x^2 e^{-\mu x^2 - 2\nu x}\, dx = -\frac{\nu}{2\mu^2} + \sqrt{\frac{\pi}{\mu^5}}\frac{(2\nu^2+\mu)}{4} e^{\nu^2/\mu}\left(1 - \text{erf}\left(\frac{\nu}{\sqrt{\mu}}\right)\right)$$

Let $t = \sqrt{\mu}x$ to obtain

$$\int_0^\infty x^2 e^{-\mu x^2 - 2\nu x}\, dx = \frac{1}{\mu^{3/2}} \int_0^\infty t^2 e^{-t^2 - 2\alpha t}\, dt,$$

with $\alpha = \nu/\sqrt{\mu}$. Completing the square gives

$$\int_0^\infty t^2 e^{-t^2 - 2\alpha t}\, dt = e^{\alpha^2} \int_0^\infty t^2 e^{-(t+\alpha)^2}\, dt.$$

The change of variables $s = t + \alpha$ gives

$$\int_0^\infty x^2 e^{-\mu x^2 - 2\nu x}\, dx = \frac{e^{\alpha^2}}{\mu^{3/2}} J(\alpha),$$

where

$$J(\alpha) = \int_\alpha^\infty s^2 e^{-s^2}\, ds - 2\alpha \int_\alpha^\infty s e^{-s^2}\, ds + \alpha^2 \int_\alpha^\infty e^{-s^2}\, ds.$$

Each of these integrals can be expressed in terms of the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds.$$

Integration by parts gives

$$\int_\alpha^\infty s^2 e^{-s^2}\, ds = \frac{\alpha}{2} e^{-\alpha^2} + \frac{1}{2} \int_\alpha^\infty,$$

the second integral evaluates by the change of variables $s^2 = u$ to be

$$\int_\alpha^\infty s e^{-s^2}\, ds = \frac{1}{2} e^{-\alpha^2}.$$

The final reduction employs the identity

$$\int_\alpha^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2} - \int_0^\alpha e^{-s^2} ds = \frac{\sqrt{\pi}}{2}(1 - \text{erf}(\alpha)).$$