

FORMULA 3.523.1

$$\int_0^\infty \frac{x^{b-1} dx}{\sinh ax} = \frac{2^b - 1}{2^{b-1} a^b} \Gamma(b) \zeta(b)$$

The integral is

$$\int_0^\infty \frac{x^{b-1} dx}{\sinh ax} = 2 \int_0^\infty \frac{x^{b-1} e^{-ax}}{1 - e^{-2ax}} dx.$$

Expanding in a geometric series

$$\int_0^\infty \frac{x^{b-1} dx}{\sinh ax} = 2 \sum_{k=0}^\infty \int_0^\infty x^{b-1} e^{-(2k+1)ax} dx.$$

The change of variables $u = (2k + 1)ax$ gives

$$\int_0^\infty \frac{x^{b-1} dx}{\sinh ax} = \frac{2\Gamma(b)}{a^b} \sum_{k=0}^\infty \frac{1}{(2k + 1)^b}.$$

To conclude, observe that

$$\sum_{k=0}^\infty \frac{1}{(2k + 1)^b} + \sum_{k=1}^\infty \frac{1}{(2k)^b} = \zeta(b).$$

Therefore,

$$\sum_{k=0}^\infty \frac{1}{(2k + 1)^b} = \frac{2^b - 1}{2^b} \zeta(b).$$