PROOF OF FORMULA 3.527.3

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{\cosh^2 ax} = \frac{4}{(2a)^\mu}(1-2^{2-\mu})\Gamma(\mu)\zeta(\mu-1)$$

The scaling $t = ax$ shows that it suffices to assume $a = 1$. Write the integral as

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{\cosh^2 x} = 4\int_0^\infty \frac{x^{\mu-1}e^{-2x}}{(1+e^{-2x})^2}\,dx.$$

Expand the integrad using

$$\frac{1}{(1+u)^2} = \sum_{k=0}^\infty (-1)^k(k+1)u^k$$

to produce

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{\cosh^2 x} = 4\sum_{k=0}^\infty (-1)^k(k+1)\int_0^\infty x^{\mu-1}e^{-2(k+1)x}\,dx.$$

The change of variables $v = 2(k+1)x$ yields

$$\int_0^\infty \frac{x^{\mu-1}\,dx}{\cosh^2 x} = -2^{2-\mu}\Gamma(\mu)\sum_{k=1}^\infty \frac{(-1)^k}{k^\mu}.$$

The last series can be expressed in terms of the Riemann zeta function by splitting the cases k even and odd to produce the identity

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^\mu} = \left(2^{1-\mu}-1\right)\zeta(\mu-1).$$

This gives the result.