PROOF OF FORMULA 3.541.10

$$\int_0^\infty e^{-qx} \frac{\sinh px}{\sinh qx} \, dx = \frac{1}{p} - \frac{\pi}{2q} \cot\left(\frac{\pi p}{2q}\right)$$

The change of variables $t = qx$ and the notation $c = p/q$ show that the evaluation is equivalent to

$$\int_0^\infty e^{-t} \frac{\sinh ct}{\sinh t} \, dt = \frac{1}{c} - \frac{\pi}{2} \cot\left(\frac{\pi c}{2}\right).$$

In order to establish this, write the integral as

$$\int_0^\infty e^{-t} \frac{\sinh ct}{\sinh t} \, dt = \int_0^\infty \frac{e^{-2t} \left(e^{ct} - e^{-ct}\right)}{1 - e^{-2t}} \, dt.$$

Entry **3.311.7** states that

$$\int_0^\infty \frac{e^{-\mu x} - e^{-\nu x}}{1 - e^{-x}} \, dx = \psi(\nu) - \psi(\mu).$$

Therefore

$$\int_0^\infty e^{-t} \frac{\sinh ct}{\sinh t} \, dt = \frac{1}{2} \left[-\psi\left(1 - \frac{c}{2}\right) + \psi\left(1 + \frac{c}{2}\right) \right].$$

The result now follows by using

$$\psi(x+1) = \psi(x) + 1/x$$

and

$$\psi(x) - \psi(1-x) = -\pi \cot \pi x.$$