PROOF OF FORMULA 3.553.1

$$\int_0^\infty \frac{\sinh^2(ax)}{\sinh x}\,\frac{e^{-x}}{x}\,dx = \frac{1}{2}\ln\left(\frac{\pi a}{\sin \pi a}\right)$$

Use the change of variables $t = 2x$ to write the integral as

$$\int_0^\infty \frac{\sinh^2(ax)}{\sinh x}\,\frac{e^{-x}}{x}\,dx = \frac{1}{2}\int_0^\infty \frac{e^{(a-1)t}(1-e^{-at})^2}{1-e^{-t}}\,\frac{dt}{t}.$$

Entry **3.413.1** states that

$$\int_0^\infty \frac{(1-e^{-ax})(1-e^{-bx})e^{-cx}}{1-e^{-x}}\,\frac{dx}{x} = \ln\left[\frac{\Gamma(c)\Gamma(a+b+c)}{\Gamma(a+c)\Gamma(b+c)}\right].$$

Employ this formula with $c = 1-a$ and $b = a$ and use the relation

$$\Gamma(a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}$$

to produce the stated answer.