PROOF OF FORMULA 3.642.1

$$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \, \cos^{2\nu-1} x}{(a^2 \sin^2 x + b^2 \cos^2 x)^{\mu+\nu}} \, dx = \frac{B(\mu, \nu)}{2a^{2\mu} b^{2\nu}}$$

Write the integral as

$$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \, \cos^{2\nu-1} x}{(a^2 \sin^2 x + b^2 \cos^2 x)^{\mu+\nu}} \, dx = \int_0^{\pi/2} \frac{\tan^{2\mu-1} x \, \sec^2 x}{(a^2 \tan^2 x + b^2)^{\mu+\nu}} \, dx.$$

Now let $r = \tan x$ to produce

$$\int_0^{\pi/2} \frac{\sin^{2\mu-1} x \, \cos^{2\nu-1} x}{(a^2 \sin^2 x + b^2 \cos^2 x)^{\mu+\nu}} \, dx = \int_0^{\infty} \frac{r^{2\mu-1} \, dr}{(a^2 r^2 + b^2)^{\mu+\nu}}.$$

The change of variables $r = bs/a$ gives

$$\int_0^{\infty} \frac{r^{2\mu-1} \, dr}{(a^2 r^2 + b^2)^{\mu+\nu}} = \frac{1}{a^{2\mu} \, b^{2\nu}} \int_0^{\infty} \frac{s^{2\mu-1} \, ds}{(s^2 + 1)^{\mu+\nu}}.$$

Finally let $t = s^2$ to see that the last integral is

$$\int_0^{\infty} \frac{s^{2\mu-1} \, ds}{(s^2 + 1)^{\mu+\nu}} = \frac{1}{2} \int_0^{\infty} \frac{s^{\mu-1} \, ds}{(1 + s)^{\mu+\nu}} = \frac{1}{2} B(\mu, \nu).$$