PROOF OF FORMULA 3.661.2

$$\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\, dx = \frac{(2n-1)!!}{(2n)!!}\, 2\pi\, (a^2+b^2)^n$$

Let $u = 2\pi - x$ to obtain

$$\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\, dx = \int_0^{2\pi} (-a\sin x + b\cos x)^{2n}\, dx.$$

Expand using the binomial theorem and add the two expression to obtain

$$2\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\, dx = 2\int_0^{2\pi} \sum_{j=0}^{n} \binom{2n}{2j} a^{2j} b^{2n-2j} \sin^{2j} x \cos^{2n-2j} x\, dx.$$

Conclude that

$$\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\, dx = 4\sum_{j=0}^{n} \binom{2n}{2j} a^{2j} b^{2n-2j} \int_0^{\pi/2} \sin^{2j} x \cos^{2n-2j} x\, dx.$$

The integral is

$$\int_0^{\pi/2} \sin^{2j} x \cos^{2n-2j} x\, dx = \frac{1}{2} B\left(j+\tfrac{1}{2}, n-j+\tfrac{1}{2}\right).$$

The result follows from

$$\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^n}(2n-1)!!,$$

and it can also be written as

$$\int_0^{2\pi} (a\sin x + b\cos x)^{2n}\, dx = \frac{2\pi}{2^{2n}} \binom{2n}{n} (a^2+b^2)^n.$$