PROOF OF FORMULA 4.224.3

$$\int_0^{\pi/2} \ln \sin x \, dx = \frac{1}{2} \int_0^{\pi} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2$$

The first identity comes from symmetry. To check the second one, the change of variables $x \mapsto \frac{\pi}{2} - x$ gives

$$\int_0^{\pi/2} \ln \sin x \, dx = \int_0^{\pi/2} \ln \cos x \, dx.$$

Therefore

$$\begin{aligned}
2 \int_0^{\pi/2} \ln \sin x \, dx &= \int_0^{\pi/2} \ln \sin x \cos x \, dx \\
&= \int_0^{\pi/2} \ln \left(\frac{\sin 2x}{2} \right) dx \\
&= -\frac{\pi}{2} \ln 2 + \int_0^{\pi/2} \ln \sin 2u \, du \\
&= -\frac{\pi}{2} \ln 2 + \frac{1}{2} \int_0^{\pi} \ln \sin v \, dv \\
&= -\frac{\pi}{2} \ln 2 + \int_0^{\pi/2} \ln \sin u \, du
\end{aligned}$$

This gives the result.