PROOF OF FORMULA 4.231.16

$$\int_0^1 \ln x \, \frac{1-x^{2n+2}}{(1-x^2)^2} \, dx = -(n+1)\frac{\pi^2}{8} + \sum_{k=1}^{n} \frac{n-k+1}{(2k-1)^2}$$

Use the expansion

$$\frac{1-x^{2n+2}}{1-x^2} = \sum_{j=0}^{n} x^{2j}$$

and let $t = x^2$ to produce

$$\begin{aligned}\int_0^1 \ln x \, \frac{1-x^{2n+2}}{(1-x^2)^2} \, dx &= \frac{1}{4} \sum_{j=0}^{n} \int_0^1 t^{j-1/2} \frac{\ln t}{1-t} \, dt \\ &= \frac{1}{4} \sum_{j=0}^{n} \sum_{m=0}^{\infty} \int_0^1 t^{m+j-1/2} \ln t \, dt.\end{aligned}$$

The change of variable $u = -\ln t$ yields

$$\int_0^1 \ln x \, \frac{1-x^{2n+2}}{(1-x^2)^2} \, dx = -\frac{1}{4} \sum_{j=0}^{n} \sum_{m=0}^{\infty} \int_0^{\infty} u e^{-(m+j-1/2)u} \, du$$

and making $v = (m+j-1/2)$ produces

$$\int_0^1 \ln x \, \frac{1-x^{2n+2}}{(1-x^2)^2} \, dx = -\sum_{j=0}^{n} \sum_{r=j}^{\infty} \frac{1}{(2r+1)^2}.$$

The result follows from here by elementary manipulations of this series and the value

$$\sum_{r=0}^{\infty} \frac{1}{(2r+1)^2} = \frac{\pi^2}{8}.$$