PROOF OF FORMULA 4.233.2

$$\int_0^1 \frac{\ln x \, dx}{1 - x + x^2} = \frac{1}{3}\left[\frac{2}{3}\pi^2 - \psi'\left(\frac{1}{3}\right)\right]$$

The integral representation of the *dilogarithm function* gives

$$\int_0^1 \frac{\ln x \, dx}{x + b} = \mathrm{Li}_2(-1/b).$$

Factoring the quadratic and denoting $r_{1,2} = \frac{1}{2}(1 \pm i\sqrt{3})$, it follows that

$$\int_0^1 \frac{\ln x \, dx}{x^2 - x + 1} = \frac{1}{r_2 - r_1}\left(\mathrm{Li}_2(-1/r_1) - \mathrm{Li}_2(-1/r_2)\right).$$

Using the series representation of the dilogarithm gives

$$\int_0^1 \frac{\ln x \, dx}{x^2 - x + 1} = -\frac{2}{\sqrt{3}}\sum_{k=1}^{\infty} \frac{\sin(\pi k/3)}{k^2}.$$

The periodicity of the values of $\sin(\pi k/3)$ and the formula

$$\psi'(x) = -\sum_{k=0}^{\infty} \frac{1}{(x+k)^2}$$

show that the series is given by

$$\psi'\left(\tfrac{1}{6}\right) + \psi'\left(\tfrac{1}{3}\right) - \psi'\left(\tfrac{2}{3}\right) - \psi'\left(\tfrac{5}{6}\right).$$

The identities

$$\psi(1-x) = \psi(x) + \pi\cot\pi x \text{ and } \psi(2x) = \tfrac{1}{2}(\psi(x) + \psi(x + \tfrac{1}{2})) + \log 2$$

are use to simplify the result.