PROOF OF FORMULA 4.236.1

$$\int_0^1 \left[\frac{1+(p-1)\ln x}{1-x} + \frac{x\,\ln x}{(1-x)^2} \right] x^{p-1}\,dx = -1 + \psi'(p)$$

The formula

$$\frac{d}{dx}\left(\frac{x\,\ln x}{1-x}\right) = \frac{1}{1-x} + \frac{\ln x}{1-x} + \frac{x\,\ln x}{(1-x)^2}$$

gives

$$\int_0^1 \left[\frac{1+(p-1)\ln x}{1-x} + \frac{x\,\ln x}{(1-x)^2} \right] x^{p-1}\,dx = \int_0^1 \left[x^{p-1}\frac{d}{dx}\left(\frac{x\ln x}{1-x}\right) + (p-2)\frac{x^{p-1}\,\ln x}{1-x} \right] dx.$$

Now use

$$\frac{d}{dx}\left[x^{p-1}\left(\frac{x\,\ln x}{1-x}\right)\right] = x^{p-1}\frac{d}{dx}\left[\frac{x\,\ln x}{1-x}\right] + (p-1)x^{p-2}\frac{x\,\ln x}{1-x}$$

to obtain

$$\int_0^1 \left[\frac{1+(p-1)\ln x}{1-x} + \frac{x\,\ln x}{(1-x)^2} \right] x^{p-1}\,dx = \int_0^1 \frac{d}{dx}\left[\frac{x^p\,\ln x}{1-x}\right] dx - \int_0^1 \frac{x^{p-1}\,\ln x}{1-x}\,dx.$$

The first integral is -1 and the second one appears from differentiating the identity

$$\psi(p) = -\int_0^1 \left(\frac{x^{p-1}}{1-x} + \frac{1}{\ln x}\right) dx.$$

This is the result.