PROOF OF FORMULA 4.241.1

$$\int_0^1 \frac{x^{2n} \ln x \, dx}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2} \left(\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} - \ln 2 \right)$$

The change of variable $t = x^2$ gives

$$\int_0^1 \frac{x^{2n} \ln x \, dx}{\sqrt{1-x^2}} = \frac{1}{4} \int_0^1 t^{n-1/2}(1-t)^{-1/2} \ln t \, dt.$$

In the proof of formula **4.253.1** it was shown that

$$\int_0^1 t^{a-1}(1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \left[\psi(a) - \psi(a+b) \right].$$

Therefore

$$\int_0^1 \frac{x^{2n} \ln x \, dx}{\sqrt{1-x^2}} = \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})}{4\Gamma(n+1)} \left[\psi(n+\tfrac{1}{2}) - \psi(n+1) \right].$$

The relations

$$\Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^n} (2n-1)!!,$$

$$\psi(n+\tfrac{1}{2}) = -\gamma + 2\sum_{k=1}^{n} \frac{1}{2k-1} - 2\ln 2$$

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n} \frac{1}{k}$$

give the result.