PROOF OF FORMULA 4.241.3

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,\ln x\,dx = \frac{(2n-1)!!}{(2n+2)!!}\,\frac{\pi}{2}\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k} - \frac{1}{2n+2} - \ln 2\right)$$

The change of variable $t = x^2$ gives

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,\ln x\,dx = \frac{1}{4}\int_0^1 t^{n-1/2}(1-t)^{1/2}\,\ln t\,dt.$$

In the proof of formula **4.253.1** it was shown that

$$\int_0^1 t^{a-1}(1-t)^{b-1}\ln t\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\left[\psi(a)-\psi(a+b)\right].$$

Therefore

$$\int_0^1 x^{2n}\sqrt{1-x^2}\,\ln x\,dx = \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{3}{2})}{4\Gamma(n+2)}\left[\psi(n+\tfrac{1}{2})-\psi(n+2)\right].$$

The relations

$$\begin{aligned}
\Gamma(n+\tfrac{1}{2}) &= \frac{\sqrt{\pi}}{2^n}\,(2n-1)!!,\\
\psi(n+\tfrac{1}{2}) &= -\gamma + 2\sum_{k=1}^{n}\frac{1}{2k-1} - 2\ln 2\\
\psi(n+2) &= -\gamma + \sum_{k=1}^{n+1}\frac{1}{k}
\end{aligned}$$

give the result.