PROOF OF FORMULA 4.244.2

$$\int_0^1 \frac{\ln x \, dx}{\sqrt[3]{1-x^3}} = -\frac{\pi}{3\sqrt{3}} \left[\ln 3 + \frac{\pi}{3\sqrt{3}} \right]$$

In the proof of formula **4.253.1** it was shown that

$$\int_0^1 t^{a-1}(1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \left[\psi(a) - \psi(a+b) \right].$$

The change of variable $t = x^3$ gives

$$\begin{aligned}
\int_0^1 \frac{\ln x \, dx}{\sqrt[3]{1-x^3}} &= \frac{1}{9} \int_0^1 t^{-2/3}(1-t)^{-1/3} \ln t \, dt \\
&= \Gamma\left(\tfrac{1}{3}\right) \Gamma\left(\tfrac{2}{3}\right) \left[\psi\left(\tfrac{1}{3}\right) - \psi(1) \right].
\end{aligned}$$

The identity

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

and the value

$$\psi\left(\frac{1}{3}\right) = -\gamma - \frac{\pi}{2\sqrt{3}} - \frac{3}{2}\ln 3 \text{ and } \psi(1) = -\gamma,$$

now give the result.