

PROOF OF FORMULA 4.244.3

$$\int_0^1 \frac{x \ln x dx}{\sqrt[3]{(1-x^3)^2}} = \frac{\pi}{3\sqrt{3}} \left(\frac{\pi}{3\sqrt{3}} - \ln 3 \right)$$

The change of variable $t = x^3$ gives

$$\int_0^1 \frac{x \ln x dx}{\sqrt[3]{(1-x^3)^2}} = \frac{1}{9} \int_0^1 t^{-1/3} (1-t)^{-2/3} \ln t dt.$$

In the proof of formula **4.253.1** it was shown that

$$\int_0^1 t^{a-1} (1-t)^{b-1} \ln t dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} [\psi(a) - \psi(a+b)].$$

Therefore

$$\int_0^1 \frac{x \ln x dx}{\sqrt[3]{(1-x^3)^2}} = \frac{2\pi}{9\sqrt{3}} \left(\psi\left(\frac{2}{3}\right) - \psi(1) \right).$$

The values

$$\psi\left(\frac{2}{3}\right) = -\gamma + \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3 \text{ and } \psi(1) = -\gamma$$

give the result.