PROOF OF FORMULA 4.245.2

$$\int_0^1 \frac{x^{4n+3}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{(2n)!!}{4(2n+1)!!}\left(\ln 2 - \sum_{k=1}^{2n+1}\frac{(-1)^{k-1}}{k}\right)$$

The change of variable $t = x^4$ gives

$$\int_0^1 \frac{x^{4n+3}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{1}{16}\int_0^1 t^n(1-t)^{-1/2}\,\ln t\,dt.$$

In the proof of formula **4.253.1** it was shown that

$$\int_0^1 t^{a-1}(1-t)^{b-1}\ln t\,dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\left[\psi(a)-\psi(a+b)\right].$$

Therefore

$$\int_0^1 \frac{x^{4n+3}\,\ln x\,dx}{\sqrt{1-x^4}} = \frac{\Gamma(n+1)\Gamma(1/2)}{16\Gamma(n+3/2)}\left[\psi(n+1)-\psi(n+3/2)\right].$$

The values

$$\begin{aligned}
\Gamma(n+\tfrac{3}{2}) &= \frac{\sqrt{\pi}}{2^{n+1}}(2n+1)!! \\
\psi(n+\tfrac{3}{2}) &= -\gamma + 2\left(\sum_{k=1}^{n+1}\frac{1}{2k-1} - \ln 2\right) \\
\psi(n+1) &= -\gamma + \sum_{k=1}^{n}\frac{1}{k},
\end{aligned}$$

give the result.