

PROOF OF FORMULA 4.261.16

$$\int_0^1 \ln^2 x \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{(2n)!!}{(2n+1)!!} \left\{ -\frac{\pi^2}{12} - \sum_{k=1}^{2n+1} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n+1} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}$$

The change of variables $t = x^2$ gives

$$\int_0^1 \ln^2 x \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{1}{8} \int_0^1 t^n (1-t)^{-1/2} \ln^2 t dt.$$

Entry **4.261.17** states that

$$\int_0^1 x^{\mu-1} (1-x)^{\nu-1} \ln^2 x dx = B(\mu, \nu) [(\psi(\mu) - \psi(\mu + \nu))^2 + \psi'(\mu) - \psi'(\mu + \nu)]$$

gives

$$\int_0^1 \ln^2 x \frac{x^{2n+1} dx}{\sqrt{1-x^2}} = \frac{1}{8} B\left(n, \frac{1}{2}\right) [(\psi(n+1) - \psi(n+3/2))^2 + \psi'(n+1) - \psi'(n+3/2)].$$

Now use

$$B(n+1, 1/2) = \frac{2(2n)!!}{(2n+1)!!}$$

and

$$\begin{aligned} \psi(n+1) &= -\gamma + \sum_{k=1}^n \frac{1}{k} \\ \psi(n+1/2) &= -\gamma + 2 \left[\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right] \\ \psi'(n+1) &= \frac{\pi^2}{6} - \sum_{k=1}^n \frac{1}{k^2} \\ \psi'(n+1/2) &= \frac{\pi^2}{2} - 4 \sum_{k=1}^n \frac{1}{(2k-1)^2} \end{aligned}$$

that appear as entries **8.365.4**, **8.366.3**, **8.366.11**, and **8.366.12**, respectively, yield the result.