PROOF OF FORMULA 4.261.18

$$\int_0^1 \frac{1-x^{n+1}}{(1-x)^2} \ln^2 x \, dx = 2(n+1)\zeta(3) - 2\sum_{k=1}^{n} \frac{n+1-k}{k^3}$$

Differentiate

$$\sum_{k=0}^{\infty} x^k = 1/(1-x)$$

to produce

$$\sum_{k=0}^{\infty} (k+1)x^k = 1/(1-x)^2.$$

Thus,

$$\begin{aligned}
\frac{1-x^{n+1}}{(1-x)^2} &= \sum_{k=0}^{\infty}(k+1)x^k - \sum_{k=0}^{\infty}(k+1)x^{k+n+1} \\
&= \sum_{k=0}^{\infty}(k+1)x^k - \sum_{k=n+1}^{\infty}(k-n)x^k \\
&= \sum_{k=0}^{n}(k+1)x^k + (n+1)\sum_{k=n+1}^{\infty} x^k.
\end{aligned}$$

Now use

$$\int_0^1 x^a \ln^2 x \, dx = \int_0^{\infty} t^2 e^{-(a+1)t} \, dt = \frac{2}{(a+1)^3},$$

to obtain

$$\begin{aligned}
\int_0^1 \frac{1-x^{n+1}}{(1-x)^2} \ln^2 x \, dx &= \sum_{k=0}^{n}(k+1)\int_0^1 x^k \ln^2 x \, dx + (n+1)\sum_{k=n+1}^{\infty}\int_0^1 x^k \ln^2 x \, dx \\
&= 2\sum_{k=0}^{n} \frac{1}{(k+1)^3} + 2\sum_{k=n+1}^{\infty} \frac{1}{(k+1)^3}
\end{aligned}$$

and this reduces to the result.