PROOF OF FORMULA 4.261.20

$$\int_0^1 \ln^2 x \, \frac{1-x^{2n+2}}{(1-x^2)^2} \, dx = \frac{7}{4}(n+1)\zeta(3) - 2\sum_{k=1}^{n} \frac{n+1-k}{(2k-1)^3}$$

Differentiate

$$\sum_{k=0}^{\infty} x^k = 1/(1-x)$$

to produce

$$\sum_{k=1}^{\infty} k x^{2k-2} = 1/(1-x^2)^2.$$

Thus,

$$\begin{aligned}
\frac{1-x^{2n+2}}{(1-x^2)^2} &= \sum_{k=1}^{\infty} k x^{2k-2} - \sum_{k=1}^{\infty} k x^{2n+2k} \\
&= \sum_{k=0}^{\infty} (k+1) x^{2k} - \sum_{k=n+1}^{\infty} (k-n) x^{2k} \\
&= \sum_{k=0}^{n} (k+1) x^{2k} + (n+1) \sum_{k=n+1}^{\infty} x^{2k}.
\end{aligned}$$

Now use

$$\int_0^1 x^a \ln^2 x \, dx = \int_0^{\infty} t^2 e^{-(a+1)t} \, dt = \frac{2}{(a+1)^3},$$

to obtain

$$\int_0^1 \ln^2 x \, \frac{1-x^{2n+2}}{(1-x^2)^2} \, dx = 2\sum_{k=0}^{n} \frac{(k+1)}{(2k+1)^3} + 2(n+1) \sum_{k=n+1}^{\infty} \frac{1}{(2k+1)^3}$$

and this reduces to the result.